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THE THEORY OF FILTRATION AT A DECREASING RATE†

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One-dimensional filtration at a rate which decreases hyperbolically, based on the Mints model [1], is considered. The system of model equations with the appropriate initial and boundary conditions is shown to be equivalent to the Goursat problems for hyperbolic equations. This is solved by the Riemann method using a method for finding a Riemann function proposed here. The method gives the well-known results for filtration at a constant rate. The hyperbolic and linear laws of filtration at a decreasing rate are shown to be equivalent under practical conditions of filter use. © 1998 Elsevier Science Ltd. All rights reserved.

The system of model equations with initial and boundary conditions corresponding to the Mints model for $\gamma = 0$ has the form

$$\rho_t + v(t)C_x = 0,\ \rho_t = \beta C - a(t)\rho \tag{1}$$

$$v(t) = v_0/(1+\gamma t),\ a(t) = a_0 v(t);\ v_0,\ \gamma,\ \alpha_0 = \text{const}$$

$$\rho|_{t=0} = 0,\ C|_{x=0} = C_0;\ C_0 = \text{const} \tag{2}$$

Here x is the coordinate along the thickness of the filter, $\upsilon(t)$ is the filtration rate, $C(x, t)$ and $\rho(x, t)$ are the required concentrations of impurities suspended in the liquid and sediment, respectively, β is a kinetic coefficient, assumed to be constant [2] and C_0 is the impurity concentration in the liquid at the filter inlet.

Eliminating the function ρ from system (1) and putting $C = U(+ \gamma t)^{-z}$, where $z = a_0\upsilon_0\gamma^{-1}$, we obtain the hyperbolic equation

$$U_{xt} + b(t)U_t - pU = 0,\ b(t) = \beta v_0^{-1}(1+\gamma t),\ p = \beta v_0^{-1}\gamma(z-1) \tag{3}$$

From conditions (2) and system (1) we obtain

$$U|_{x=0} = C_0(1+\gamma t)^z,\ U|_{t=0} = C_0\exp(-\beta v_0^{-1}x) \tag{4}$$

Problem (3), (5) is a special case of the Goursat problem [3], which is solved simply by finding the corresponding Riemann function R. By the method of determining the Riemann function that we propose here, for the general second-order linear equation of hyperbolic type with two independent variables, as it applies to Eq. (3), we will have

$$R = \exp[b(t)(x-\xi)]\sum_{n=0}^{\infty}\frac{T_n^n}{n!}(x-\xi)^n \tag{5}$$

$$T_n^n = \int_\eta^t [p-(n-1)b_t(t)]\int_\eta^t \ldots \int_\eta^t [p-b_t(t)]\int_\eta^t p(dt)^n$$

Substituting the expressions for $b(t)$ and p into (5) we obtain

$$R = \exp[(1+\tilde{t})(\tilde{x}-\tilde{\xi})]\sum_{n=0}^{\infty}\binom{z-1}{n}\frac{[(\tilde{t}-\tilde{\eta})(\tilde{x}-\tilde{\xi})]^n}{n!} =$$

$$= \exp[(1+\tilde{t})(\tilde{x}-\tilde{\xi})]_1F_1[-(z-1),1;-(\tilde{t}-\tilde{\eta})(\tilde{x}-\tilde{\xi})] \tag{6}$$

$$(\tilde{t}-\tilde{\eta}) = \gamma(t-\eta),\ (\tilde{x}-\tilde{\xi}) = \frac{\beta}{v}(x-\xi)$$

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Here η and ξ are the current values of x and t respectively and ${}_1F_1$ is the confluent hyperbolic function.
Using the relation [4]

$$ {}_1F_1(\alpha,\gamma;z) = e^z {}_1F_1(\gamma-\alpha,\gamma;-z) $$

we can represent the Riemann function in the form

$$ R = \exp[(\tilde{x}-\tilde{\xi})(\tilde{\eta}+1)]_1 F_1(z,1;(\tilde{t}-\tilde{\eta})(\tilde{x}-\tilde{\xi})] \tag{7} $$

Then, from expression (6) using the Riemann method [3] we find

$$ C(\tilde{x},t) = \frac{C_0}{(1+\tilde{t})^z}\{e^{-\tilde{x}}{}_1F_1[-(z-1),1;-\tilde{x}\tilde{t}] + zG(x,t,z-1)\} \tag{8} $$

$$ G(\tilde{x},\tilde{t},z) = \int_0^{\tilde{t}} (1+\tau)^z e^{-\tilde{x}(1+\tau)}{}_1F_1[-z,1;\tilde{x}(\tau-\tilde{t})]d\tau $$

We will discuss some features of this solution. If the quantity $z-1$ is equal to a non-negative integer n, the confluent hypergeometric series terminates, and [4]

$$ {}_1F_1(-n,1;-y) = L_n(-y) \tag{9} $$

where L_n is a Laguerre polynomial of degree n. We also have the following limit [4]

$$ \lim_{n\to\infty} L_n\left(\frac{y}{n}\right) = J_0\left(2\sqrt{y}\right) \tag{10} $$

Here J_0 is a zero-order Bessel function of the first kind.
Thus for sufficiently large n the function ${}_1F_1$ in expression (9) can be replaced by a zero-order Bessel function of the first kind of imaginary argument I_0. In fact

$$ \lim_{n\to\infty} L_n[-\tilde{x}(\tilde{t}-\tau)] = \lim_{n\to\infty} L_n\left[-\frac{\tilde{x}(\tilde{t}_1-\theta)}{n}\right] = I_0\left(2\sqrt{\tilde{x}(\tilde{t}_1-\theta)}\right),\ t_1 = a_0\nu_0 t \tag{11} $$

Thus in this case expression (8) takes the form

$$ C(\tilde{x},\tilde{t}_1) = C_0 e^{-\tilde{x}-\tilde{t}_1}\left\{I_0\left(2\sqrt{\tilde{x}\tilde{t}_1}\right) + H(\tilde{x},\tilde{t}_1)\right\},\ H(\tilde{x},\tilde{t}_1) = \int_0^{\tilde{t}_1} e^{\theta} I_0\left(2\sqrt{\tilde{x}(\tilde{t}_1-\theta)}\right)d\theta \tag{12} $$

Changing to the variable $\varphi = \tilde{x}(\tilde{t}_1-\theta)$, we have the solution obtained by Tikhonov [5], corresponding to a constant filtration rate ($\gamma = 0$). However, since (12) is an asymptotic representation of the solution (8) as $z = a_0\upsilon_0\gamma^{-1} \to \infty$, we can conclude that it also holds for $a_0\upsilon_0 \gg \gamma$, that is, at relatively high filtration rates when the sediment is unstable, and the probability that the liquid flow will detach sediment particles is high.
A similar method is used to find the concentration ρ. Eliminating the function C from system (1) and putting $\rho = V(1+\gamma t)^{-z}$, we obtain an equation in $V(x,t)$ which differs from Eq. (3) in that $p = a_0\beta$. Transforming conditions (2) using system (1), we obtain

$$ V|_{t=0} = 0,\ V|_{x=0} = \frac{\beta C_0}{\gamma(1+z)}[(1+\gamma t)^{1+z}-1] \tag{13} $$

In this case too the Riemann function is found from formula (5), but with $p = a_0\beta$, resulting in a formula similar to (6) but with $z-1$ replaced by z. Using the Riemann method [3], we find

$$ \rho(\tilde{x},\tilde{t}) = \frac{\beta C_0}{\gamma(1+\tilde{t})^z}G(\tilde{x},\tilde{t},z) \tag{14} $$

If z is a sufficiently large integer then, according to relation (11), the expression

$$ \rho(\tilde{x},\tilde{t}_1) = \frac{\beta C_0}{a_0\nu_0}e^{-\tilde{x}-\tilde{t}_1}H(\tilde{x},\tilde{t}_1) \tag{15} $$

can be used instead of (14).

Solutions (8) and (14) and their asymptotic representations (12) and (15) have a specific statistical interpretation, since system (1) reduces to stochastic equations of the Kolmogorov–Feller type in the concentrations C and ρ. In particular, it has been shown [6] that the Mints model [1], which is the prototype of system (1) with $\gamma = 0$, can be reduced to the Kolmogorov–Feller equations; solution (15) for the concentration ρ is identical to the Rayleigh–Rice integral distribution function and can be represented in the form of arithmetic operations on the Poisson probabilities of suspended particles of impurity being captured and of particles of the sediment which forms being detached.

Note that solutions (8) and (14) give a good approximation for a linearly decreasing filtration rate. In practice, by the time t_* at which filters cease to offer protection, over a wide range of conditions of filter use, the rate has fallen by no more than 10–15%. This means that

$$v(t)=v_0/(1+\gamma t)\approx v_0(1-\gamma t) \quad \text{for} \quad t\leqslant t_*$$

that is, with the given constraints on the filtration rate, the hyperbolic and linear laws of rate variation can be considered to be equivalent.

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